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The unsteady evolution of the singularity at separation

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SUMMARY

The unsteady boundary layer equations are studied close to a separation point. A perturbation procedure, based on an idea of Kaplun, leads to a nonlinear integro-differential equation for the skin-friction. A uniformly valid small-time analysis of this equation shows how an initial square-root singularity at a separation point is smoothed out by unsteady processes.

1. Introduction

It has been firmly established (Goldstein [6]; Stewartson [10]; Terrill [13]) that in general, the two-dimensional, incompressible steady boundary layer in an adverse pressure gradient has singular behaviour at the separation point. Thus Goldstein [6] and Stewartson [10] derived a coordinate expansion to describe the flow near separation, showing that the skin-friction vanishes like $(x_s-x)^{\frac{1}{2}}$. Terrill [13], who generalized the result to include the effects of wall suction, confirmed the validity of the expansion by careful comparison with the results of a numerical integration of the boundary layer equations. Now reasonable men may differ in their opinion as to whether this singular behaviour occurs in practice. It may well be that the interaction of the boundary layer with the inviscid free stream is such as to eliminate the singular behaviour at separation. But whether the singularity occurs or not, it surely plays a fundamental role in the question of separation. For if it does exist then it has far-reaching effects on any attempt to integrate the boundary layer equations through the separation point; and if the pressure gradient adjusts itself precisely to ensure regular behaviour then this is an important restraint on the inviscid flow. With regard to this, no such restraint would exist if the equations never admitted singular behaviour, so that whatever point of view one takes it is important to decide whether or not singular behaviour is possible for a given boundary layer system. This is the basic motivation of recent attempts to describe the flow near separation for a boundary layer with blowing (and zero pressure gradient), Catherall, Stewartson, Williams [5]; the compressible boundary layer, Buckmaster [1], [2]); the slender axisymmetric boundary layer, Buckmaster [3]); and the three-dimensional boundary layer, Buckmaster ([4]).

In this paper the *unsteady* separated boundary layer is discussed, not because it is anticipated that any results derived can be confirmed experimentally, but because of the light that they may throw on the mathematical theory of separation. There are several problems of interest. Imagine, for example, that we have a steady separated flow with its attendant square-root singularity. If now the flow is perturbed, the point of zero skinfriction will move and we can envisage a situation in which this point ultimately settles down at a position removed from its original location. The question is, what happens to the singularity between the times $t=0$ and $t=\infty$? It is conceivable that the initial singularity is swept away and a new one emerges only in the limit. Alternatively, the singularity might retain some kind of identity, moving in some fashion from one position to the other. A description along the latter lines has been attempted by Telionis [12] (see also Sears and Telionis [9]), in which it was supposed that the singularity moves as a square-root into the interior of the boundary layer, only reattaching itself to the wall in the limit $t\rightarrow\infty$. Unfortunately there is no numerical evidence to support this point of view, although it must be admitted that singular behaviour has been numerically lost before. But in the absence of such evidence a high degree of mathematical self-consistency is required if such a model is to appear plausible, and Telionis and Sears were unable to develop the theory very far in this respect. Consequently it can not be claimed that the problem is now understood.

A somewhat different problem is that of a flow impulsively generated from rest. On an impulsively moved circular cylinder, for example, the boundary layer can be expected to remain thin for all finite time, and separation in the sense of breakdown, with its attendant singularity, will only emerge in the limit $t \to \infty$. In such a flow the concept of a moving singularity plays no role. An understanding of how the singularity emerges could lead to a greater understanding of how the boundary layer first interacts in a significant way with the outer inviscid flow, a classical problem in the theory of separation.

A start on understanding both of these problems might well be possible if we are willing to confine our attention to a small neighborhood of the point where the skin-friction vanishes, x_0. For close to such a point the boundary-layer flow will be a small perturbation about $\psi \sim y^3$, where ψ is the stream function and y is the distance measured normal to the wall; and this affords a substantial simplification. In applying this idea to the problem mentioned above, the perturbation can be appropriate at all times provided x_0 moves only a small distance. If the disturbance generating the unsteadiness is too large for this to be true, the perturbation is only appropriate during a small time interval, or in a neighborhood of $t = \infty$. It is then capable of describing what happens for small times when the initial singularity, if present, is modified; or for large times, as the final singularity emerges. A central result of the present paper is the derivation of an equation capable of describing such changes in the skin-friction. Unfortunately we have been unable to deduce the large time asymptotics of this equation, and so the analysis is restricted to small times.

2. Perturbation analysis

The boundary layer equations to be considered here are,

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{\partial p}{\partial x} + \frac{\partial^2 u}{\partial y^2} \quad \left(u = \frac{\partial \psi}{\partial y} \quad v = -\frac{\partial \psi}{\partial x}\right) \tag{2.1}$$

In order to discuss the small time evolution of the separation singularity we imagine that at $t = 0$ the flow corresponds to a steady separated boundary layer. Confining our attention to the vicinity of the initial separation point, for appropriate disturbances, the flow will be a perturbation about $\psi \sim y^3$ for small time. (An impulsive change in streamwise velocity is not appropriate, because of the associated vortex sheet). A consideration of all possible perturbations should exhaust the behavioral possibilities for the boundary layer. Of course, for our purposes it is not necessary to consider *all* possible perturbations: special ones will suffice provided they are typical. Now in general a perturbation procedure will lead to a sequence of linear equations. Exceptions to this occur when the solution at any stage contains an arbitrary function (eigensolution) to be determined by consideration of later stages. This eigensolution can then satisfy a nonlinear equation from which singular behaviour can emerge. Clearly it is this kind of perturbation that we are interested in. Kaplun considered perturbations of this kind to study the steady two-dimensional incompressible boundary layer, and Buckmaster used the same idea to study compressible and three-dimensional boundary layers. These applications reveal that Kaplun's method can be very useful, and we use it here.*

* It has been suggested, in private communications to the author, that Kaplun's method is only suggestive and that consequently one should always, as a preference, seek a coordinate expansion of the Goldstein variety. It is certainly true that Kaplun's method is only suggestive, and a coordinate expansion does sometimes have some advantages, but the author can not accept that a coordinate expansion is any less suggestive than Kaplun's method. In both cases one can never be sure that all possible perturbations have been exhausted. (Thus Stewartson, [11] using a coordinate expansion, concluded that heat transfer eliminates the separation singularity, whereas the true answer, at least for a cold wall, is that terms unsuspected by Stewartson are required in the expansion.) And in both cases there is no guarantee that the expansions mean anything, since nothing is known regarding their convergence or asymptotic nature.

We start with a brief description of Kaplun's method applied to the steady-state problem. With the unsteady term dropped and the pressure gradient taken as 1, the boundary layer is described by

$$\frac{\partial\psi}{\partial y}\cdot\frac{\partial^2\psi}{\partial x\,\partial y}-\frac{\partial\psi}{\partial x}\cdot\frac{\partial^2\psi}{\partial y^2}=-1+\frac{\partial^3\psi}{\partial y^3}. \tag{2.2}$$

A solution is sought satisfying the following initial conditions,

$$\psi(0,y)=\tfrac{1}{6}y^3+\varepsilon y^2+\varepsilon^2 f(y)+O(\varepsilon^3) \tag{2.3}$$

together with the boundary conditions

$$\psi(x,0)=\frac{\partial\psi}{\partial y}(x,0)=0\,. \tag{2.4a}$$

The usual matching with the free stream as $y\to\infty$ can not be imposed since the profile at $x=0$ violates it. This is not surprising since the perturbation procedure is only valid near the wall. Instead, we use

$$\lim_{y\to\infty} y^{-n}\psi=0 \quad \text{for} \quad \text{some } n \tag{2.4b}$$

which is equivalent to insisting that ψ diverges at most algebraically for large y.

The ε that appears in (2.3) is an artificially introduced small parameter that generates perturbations about the zero skin-friction $O(1)$ flow.

A solution for ψ is sought in the form

$$\psi\sim\tfrac{1}{6}y^3+\varepsilon\psi_1+\varepsilon^2\psi_2+O(\varepsilon^3)$$

which leads to the following equation for ψ_1,

$$\tfrac{1}{2}y^2\frac{\partial^2\psi_1}{\partial x\,\partial y}-y\frac{\partial\psi_1}{\partial x}=\frac{\partial^3\psi_1}{\partial y^3} \tag{2.5}$$

This equation does not have an acceptable solution for arbitrary initial data, but with the choice (2.3),

$$\psi_1=y^2F(x)\quad F(0)=1$$

where $F(x)$ is otherwise an arbitrary function. Note that the $O(\varepsilon)$ skin-friction is $2F$. It should not be thought that the dominance of ψ_1 over ψ_0 for small y implies a nonuniformity at the wall. All the higher order terms also behave like y^2 near the wall, so that each of them satisfies the wall boundary condition.

Turning now to the equation for $\bar\psi_2(p,y)$, the x Laplace Transform of ψ_2,

$$\frac{d^3\bar\psi_2}{dy^3}-\frac{p}{2}y^2\frac{d\bar\psi_2}{dy}+py\bar\psi_2=yf(y)-\tfrac{1}{2}y^2f'(y)+2y^2\,\overline{F\frac{dF}{dx}}$$
$$\equiv P(y) \tag{2.6}$$

and this has a solution satisfying the conditions (2.4) if and only if (Buckmaster [2]),

$$\int_0^\infty ds\,s^{\frac{3}{2}}K_{\frac{1}{4}}\left(\frac{s^2}{2\sqrt{2}}\right)P\left(\frac{s}{p^{\frac{1}{4}}}\right)=0 \tag{2.7}$$

This yields an equation for $F(x)$ once $f(y)$ is chosen. For example, if $f(y)\sim y^n$ then

$$F=[1-Cx^{\frac{1}{4}(n-1)}]^{\frac{1}{2}}$$

where C is a constant. If C is positive, this describes separation with a square-root singularity.

It is important to notice that a different choice of $f(y)$ gives rise to a different value of F. Consequently, in an unsteady problem, if f is changed in time then F must also change, and if f approaches a new steady state so must F, and both the initial and final solutions for F will have singularities. This is the kind of perturbation we shall consider to describe the unsteady evolution of the singularity. It has the overwhelming advantage that it leads to a uniformly valid (in x) description of the $O(\varepsilon)$ skin-friction. This simplification would not occur if, for example, we perturbed the pressure gradient. In terms of application to real boundary layer situations, the equation that we derive correctly describes the flow near the point of zero skin-friction, $x_0(t)$, as this moves under the influence of some upstream disturbance. This description is valid for all time if the movement is so small that at the instantaneous location of x_0 the flow has always been a perturbation about $\psi \sim y^3$. If the disturbance is too large for this to be true the equation is valid only for a small time interval, in particular for small times when the initial singularity is being modified, or for large times when the final singularity is emerging. As far as the emergence of the singularity at large times is concerned, the essential features will not depend on the initial data, and so the interpretation is not restricted to the special disturbance considered here.

A distinctive feature of the perturbation problem is that an $O(\varepsilon)$ flow (ψ_1) develops as a consequence of $O(\varepsilon^2)$ data. This suggests that the appropriate time variable is $\tau = \varepsilon t$, and then the equation for ψ_1 has an eigensolution. Accordingly, the correct formulation of the problem is,

$$\varepsilon \frac{\partial^2 \psi}{\partial y \partial \tau} + \frac{\partial \psi}{\partial y} \cdot \frac{\partial^2 \psi}{\partial x \partial y} - \frac{\partial \psi}{\partial x} \frac{\partial^2 \psi}{\partial y^2} = -1 + \frac{\partial^3 \psi}{\partial y^3} \tag{2.8}$$

$$\psi \sim \tfrac{1}{6} y^3 + \varepsilon \psi_1(x, y; \tau) + \varepsilon^2 \psi_2(x, y; \tau) + \ldots$$

ψ_1 satisfies (2.5) so that we write its solution as

$$\psi_1 = y^2 F(x, \tau) \tag{2.9}$$

where it is assumed that the conditions at $x=0$ and $\tau=0$ are consistent with this.

The equation for $\bar{\psi}_2$, the Laplace Transform in x of ψ_2, is

$$\frac{d^3 \bar{\psi}_2}{dy^3} - \frac{p}{2} y^2 \frac{d\bar{\psi}_2}{dy} + py\bar{\psi}_2 = P(y)$$

$$P(y) \equiv y\psi_2(0, y; \tau) - \tfrac{1}{2} y^2 \frac{\partial \psi_2}{\partial y}(0, y; \tau) + 2y \frac{\partial \bar{F}}{\partial \tau} + 2y^2 \overline{F \frac{\partial F}{\partial x}} \tag{2.10}$$

The integral restraint (2.7) now leads to an equation for F, the unknown $O(\varepsilon)$ skin-friction. Choose as initial (in x) conditions,

$$F(0, \tau) = 1 \tag{2.11}$$

$$\psi_2(0, y; \tau) = c(\tau) y^n$$

and then the integral restraint leads to

$$0 = \left(1 - \frac{n}{2}\right)\left(\frac{8}{p}\right)^{n/4} \int_0^\infty dm\, m^{n+\frac{5}{2}} K_{\frac{1}{4}}(m^2) c(\tau)$$

$$+2 \int_0^\infty dm\, m^{\frac{5}{2}} K_{\frac{1}{4}}(m^2) \frac{\partial \bar{F}}{\partial \tau} + 2\left(\frac{8}{p}\right)^{\frac{1}{4}} \int_0^\infty dm\, m^{\frac{7}{2}} K_{\frac{1}{4}}(m^2) \overline{F \frac{\partial F}{\partial x}}. \tag{2.12}$$

This may be inverted as it stands, or after first multiplying by $p^{\frac{1}{4}}$. The resultant equations are equivalent, but sometimes one form is more useful than the other.

Now imagine that for all negative time we have the steady state solution corresponding to

$$c(\tau) = \frac{\Gamma(\frac{5}{4})}{\left(1 - \frac{n}{2}\right) \Gamma\left(1 + \frac{n}{4}\right) 2^{5(n-1)/4}}$$

namely,

$$F(x) = (1 - x^{\frac{1}{4}(n-1)})^{\frac{1}{2}}$$

At $\tau = 0$, $c(\tau)$ is impulsively changed to a new constant value, so that with the choice $n = 9$, the equation for F can be written in the following equivalent forms for $\tau > 0$

$$0 = Bx^2 + F^2 - 1 + \frac{1}{\Gamma(\frac{5}{4})\, 2^{\frac{1}{4}}} \int_0^x d\tilde{x}\, \frac{\frac{\partial F}{\partial \tau}(\tilde{x}, \tau)}{(x - \tilde{x})^{\frac{1}{4}}}$$

$$0 = B\, \frac{8^{\frac{5}{4}}}{5\sqrt{2}} + \Gamma(\tfrac{3}{4})\, \frac{\partial F}{\partial \tau} + \frac{1}{2^{\frac{3}{4}}} \int_0^x d\tilde{x}\, \frac{F(\tilde{x}, \tau)\, \frac{\partial F}{\partial \tilde{x}}(\tilde{x}, \tau)}{(x - \tilde{x})^{\frac{3}{4}}}$$

$$F(x, 0) = (1 - x^2)^{\frac{1}{2}} \quad F(0, \tau) = 1 \tag{2.13}$$

B is an arbitrary, assigned, constant related to the choice of $c(\tau)$ for $\tau > 0$. The ultimate steady state corresponding to (2.13) is

$$F = (1 - Bx^2)^{\frac{1}{2}}$$

so that the choice of B is equivalent to locating the ultimate separation point. The separation point moves upstream from its original position if $B > 1$, downstream if $B < 1$.

The solution of (2.13) is a formidable problem and the only hope seems to be in a small or large-time analysis. A large time analysis would be of particular interest for what it might tell us about the initial interaction of the boundary layer with the freestream, but has so far eluded solution. Thus, the discussion will be limited to a small time analysis. It should be noted that, since the problem has been formulated only with upstream data, the solution is presumably only meaningful ahead of any region of reversed flow.

3. Small time analysis

In this section the small time solution of (2.13) is considered, to describe what happens to the initial singularity. Three regions are needed to describe the flow, one at the origin, one for intermediate values of x, and one in the neighborhood of $x = 1$. The singular behaviour near the origin is merely a creature of our formulation, and so we do not discuss it. Actually, we are only interested in the behaviour near $x = 1$, but the easiest way to deduce this is to first construct the intermediate solution.

The second of equations (2.13) is the appropriate form for discussing the intermediate solution when $x = O(1)$, $(1 - x) = O(1)$. We seek a solution in the form

$$F \sim (1 - x^2)^{\frac{1}{2}} + \sum_{n=1}^{\infty} \tau^n F_n(x) \tag{3.1}$$

Then

$$F_1 = c_1 x^{\frac{5}{4}} \quad c_1 \equiv \frac{8.2^{\frac{1}{4}}(1 - B)}{5\Gamma(\frac{3}{4})}$$

As expected, if $B < 1$ so that the separation point moves downstream, the skin-friction increases with time.

None of the higher order terms are so simple, and furthermore, for small x, $F_2 \sim x^{\frac{1}{2}}$, $F_3 \sim x^{-\frac{1}{4}}$, and F_4 is not even defined by this procedure since the relevant integrals do not exist. The reason for this is that at each stage the integral in (2.13) is computed using the intermediate expansion, which is not valid for very small x. The way to handle the difficulty is to use the small x solution with the intermediate solution to form a composite, uniformly valid, expansion with which the integral can be correctly evaluated. It turns out that this is equivalent to ignoring the small x solution, and retaining only the finite part of the integral to determine each of the F_n.

F_2 and higher terms in the intermediate expansion are singular at $x = 1$, heralding the fact that in a neighborhood of the initial singularity a different kind of expansion is required. An examination of the integral equation shows that, as $x \to 1$,

$$
\begin{aligned}
&F_{2n} \sim (1-x)^{\frac{1}{2}-(3n/4)} && n=0, 1, \ldots \\
&F_{2n+1} \sim (1-x)^{\frac{1}{4}-(3n/4)} && n=1, 2, \ldots
\end{aligned}
\tag{3.2}
$$

so that, omitting (constant) coefficients

$$F \sim (1-x)^{\frac{1}{2}} + \tau + \tau^2 (1-x)^{-\frac{1}{4}} + \tau^3 (1-x)^{-\frac{1}{2}} + \tau^4 (1-x)^{-1} + \tau^5 (1-x)^{-\frac{5}{4}} + \ldots$$

as $x \to 1$.

This implies that the appropriate variable in the separation region is

$$\varphi \equiv \frac{\tau}{(1-x)^{\frac{3}{8}}} \tag{3.3}$$

and then the expansion in the separation region will have the form,

$$F \sim c_1 \tau + \tau^{\frac{4}{3}} h_1(\phi) + \tau^{\frac{5}{3}} h_2(\phi) + \ldots \tag{3.4}$$

The term $\tau^{\frac{4}{3}} h_1(\phi)$ is contained in

$$F_0 + \tau^2 F_2 + \tau^4 F_4 + \ldots$$

where just the leading approximation to each term is retained as $x \to 1$. An examination of the integral equation shows that

$$
\begin{aligned}
&F_0 \sim \sqrt{2}(1-x)^{\frac{1}{2}} \\
&0 \sim nF_n + \frac{c_1}{2^{\frac{3}{4}} \Gamma(\frac{3}{4})} \int_0^x d\tilde{x} \, \frac{F'_{n-2}(\tilde{x})}{(x-\tilde{x})^{\frac{3}{4}}} \qquad n=2, 4, \ldots
\end{aligned}
\tag{3.5}
$$

where just the leading term of the integral is retained. Consequently, if we write

$$F_n \sim D_n (1-x)^{\frac{1}{2}-(3n/8)}$$

then $D_0 = \sqrt{2}$ and (3.5) yields the recurrence relation

$$\frac{D_n}{D_{n-2}} = -\frac{3 \cdot c_1}{8 \cdot 2^{\frac{3}{4}}} \left(1 - \frac{10}{3n}\right) \frac{\Gamma\left(\frac{1}{4}\right) \Gamma\left(\frac{3n}{8} - \frac{1}{2}\right)}{\Gamma\left(\frac{3}{4}\right) \Gamma\left(\frac{3n}{8} - \frac{1}{4}\right)} \tag{3.6}$$

In addition,

$$h_1(\phi) = \phi^{-\frac{4}{3}} \sum_{n=0, 2, \ldots}^{\infty} D_n \phi^n \tag{3.7}$$

There are two things that should be checked. First, that the series defined by (3.6, 3.7) defines a function for all values of ϕ; and second, that $h_1(\phi)$ is bounded on the positive real axis when the separation point moves downstream ($\phi \to +\infty$ is then an allowable limit). The first check is easily disposed of since, for large n, the ratio of successive coefficients is proportional to $n^{-\frac{1}{4}}$, and therefore the series defines an entire function. The boundedness question can be settled by summing the series.

We have

$$
\begin{aligned}
&h_1(\phi) = -\frac{2^{\frac{3}{2}}}{3\Gamma(\frac{1}{2})} \phi^{-\frac{4}{3}} \sum_{n=0}^{\infty} (-1)^n \left(\frac{4\gamma}{3}\right)^n \frac{\Gamma(\frac{3}{4}n + \frac{1}{2})}{(n-\frac{2}{3}) n!} \phi^{2n} \\
&\gamma \equiv \frac{3}{8} \frac{\Gamma(\frac{1}{4})}{2^{\frac{3}{4}} \Gamma(\frac{3}{4})} c_1
\end{aligned}
\tag{3.8}
$$

Representing the Gamma function by the usual integral

$$h_1(\phi) = -\frac{2^{\frac{3}{2}}}{3\Gamma(\frac{1}{2})}\phi^{-\frac{4}{3}}\int_0^\infty dt\, e^{-t}t^{-\frac{1}{2}}\sum_{n=0}^{\infty}(-1)^n\frac{z^n}{(n-\frac{2}{3})n!}$$

$$z \equiv \frac{4\gamma}{3}\, t^{\frac{3}{4}}\phi^2 \tag{3.9}$$

Since,

$$\sum_{n=0}^{\infty}(-1)^n\frac{z^n}{(n-\frac{2}{3})n!} = -\tfrac{3}{2}e^{-z} - \frac{3z}{2}\int_0^1 dt\, t^{-\frac{2}{3}}e^{-zt}$$

it follows that,

$$h_1(\phi) = \frac{2^{\frac{1}{2}}}{\Gamma(\frac{1}{2})}\phi^{\frac{8}{3}}\int_0^\infty dt\, \exp\left[-\phi^8\left(t + \frac{4\gamma}{3}t^{\frac{3}{4}}\right)\right]\left[t^{-\frac{1}{2}}+\gamma t^{-\frac{3}{4}}\right]. \tag{3.10}$$

The behaviour of $h_1(\phi)$ for large positive ϕ depends on the sign of γ, which has the same sign as c_1. If c_1 is positive, the skin-friction increases with time, the separation point moves downstream, and the solution is valid in $0 \leqq x \leqq 1$. In other words, ϕ is allowed to assume arbitrarily large values and the expression (3.10) must then be bounded. On the other hand if γ (and c_1) are negative, the separation point moves upstream and the solution is only valid in $0 \leqq x \leqq x_0(\tau)$. The maximum value of ϕ is then $\tau[1-x_0(\tau)]^{-\frac{3}{8}}$, and we shall see that this is small for small τ. No boundedness condition on $h_1(\phi)$ is then needed.

Consider the case $\gamma > 0$. Then for large ϕ the major contribution to the integral in (3.10) comes from the origin. The asymptotic behaviour can then be determined by introducing a new variable

$$s = t + \frac{4\gamma}{3}t^{\frac{3}{4}}$$

and expanding $t^{-\frac{1}{2}}(s)$ for small s, hence

$$h_1(\phi) \sim 2^{\frac{1}{2}}\frac{\Gamma(\frac{1}{3})}{\Gamma(\frac{1}{2})}\left(\frac{4\gamma}{3}\right)^{\frac{2}{3}} + O(\phi^{-\frac{8}{3}}) \qquad \gamma > 0 \tag{3.11}$$

which is bounded, as required.

If γ is negative, the major contribution to the integral comes from the neighborhood of $t = \phi^4$. The asymptotic analysis then leads to an expression that is unbounded for large ϕ, but this does not matter. For examination of the intermediate expansion (3.1), when c_1 is negative, shows that

$$1 - x_0(\tau) \sim \frac{c_1^{\,2}}{2}\tau^2 \tag{3.12}$$

so that ϕ maximum is $O(\tau^{\frac{1}{4}})$, and therefore small. In other words the intermediate expansion may be used all the way to separation when the separation point moves forward. Note that (3.12) implies that when $x_0(\tau)$ moves forward under the influence of an upstream disturbance, its' initial velocity varies linearly with time. It might be possible to verify this experimentally.

The normal component of velocity, v, is unbounded at $\tau = 0$. For finite times we have,

$$v_1 = -y^2\frac{\partial F}{\partial x} \sim -\frac{3}{8}\frac{y^2}{\tau^{\frac{4}{3}}}\phi^{11/3}h_1'(\phi)$$

in the separation region. It follows that, when the separation point moves downstream, $v_1(1, y, \tau) \sim y^2\tau^{-\frac{4}{3}}$. On the other hand, if we follow the point of zero skin-friction as it moves upstream, the intermediate expansion implies $v_1(x_0, y, \tau) \sim y^2\tau^{-1}$.

The above results, which describe the smoothing out of the initial singularity, do not contradict the work of Telionis and Sears. The present description does not extend into regions of reversed flow, so that it may be concluded that if the singularity survives, it must be located in such a region. It might be possible to resolve the question by extending the present analysis to include regions of reversed flow. Upstream data would have to be given for fluid travelling downstream and vice-versa.

A more interesting problem, perhaps, is the question of what happens for large times. A classical problem in the theory of separation is the large time behaviour of the boundary layer on an impulsively started bluff body. Proudman and Johnson [8] have shown that, at a rear stagnation point, the boundary layer eventually thickens exponentially in time. It is not known what happens in the remainder of the reversed flow region. The formulation of Section 2 is capable of describing the emergence of a singularity, and therefore something of the behaviour of the displacement thickness at the upstream boundary of the reversed flow region. In respect to this we repeat an earlier observation that the late time behaviour should not depend critically on the initial data i.e. on the special data considered here. Note that we could easily choose initial data for F that does not have a singularity.

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REFERENCES

[1] J. Buckmaster, The behavior of a laminar compressible boundary layer on a cold wall near a point of zero skin-friction, *J. Fluid Mech.*, 44 (1970) 237.
[2] J. Buckmaster, Separation and the Compressible Boundary Layer, *J. Eng. Math.*, 5 (1971) 71.
[3] J. Buckmaster, Effect of Transverse Curvature on the Singularity at Separation for a Laminar Boundary Layer, *Physics of Fluids* 14 (1971) 1593.
[4] J. Buckmaster, Perturbation Technique for the Study of Three-Dimensional Separation, *Physics of Fluids*, 15 (1972) 2106.
[5] D. Catherall, K. Stewartson, and P. G. Williams, Viscous flow past a plate with uniform injection, *Proc. Roy. Soc.* A 284 (1965) 370.
[6] S. Goldstein, On Laminar Boundary-Layer Flow, *Quart. J. Mech. Appl. Math.* 1 (1948) 43.
[7] S. Kaplun in *Fluid Mechanics and Singular Perturbations* edited by P. A. Lagestrom, L. N. Howard, and C. S. Liu (Academic, New York, 1967) p. 145.
[8] I. Proudman, and K. Johnson, Boundary layer growth near a rear stagnation point, *J. Fluid Mech.*, 12 (1962) 161.
[9] W. R. Sears and D. Telionis, Unsteady Boundary-Layer Separation, To appear in the *Proceedings of the* 1971 *I.U.T.A.M. Symposium on Unsteady Boundary Layers.*
[10] K. Stewartson, On Goldstein's Theory of Laminar Separation, *Quart. J. Mech. Appl. Math.*, 11 (1958) 399.
[11] K. Stewartson, The behaviour of a laminar compressible boundary layer near a point of zero skin-friction, *J. Fluid Mech.*, 12 (1962) 117.
[12] D. Telionis, *Unsteady Boundary-Layer Separation*, Ph. D. Thesis, Cornell University, School of Engineering (1970).
[13] R. M. Terrill, Laminar Boundary-Layer Flow Near Separation with and without Suction, *Phil. Trans.*, A 253 (1960) 55.